Geometric Nonlinear Analysis of Composite Beams Using Wiener-Milenković Parameters

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This paper presents a geometric nonlinear analysis of composite beams, and includes static, dynamic, and eigenvalue analyses. The increase in size and flexibility of engineering components such as wind turbine blades causes geometric nonlinearity to play an increasingly significant role in structural analysis. The Geometric Exact Beam Theory (GEBT), pioneered by Reissner and extended by Hodges, is adopted as the foundation for this work. Special emphasis is placed on the vectorial parameterization of finite rotation, which is a fundamental aspect in the geometric nonlinear formulation. This method is introduced based on the Euler’s rotation theorem and the property of rotation operation: length preservation of the rotated vector. The GEBT is then implemented with the Wiener-Milenković parameters using a mixed formulation. Several numerical examples are studied based on the derived theory and the results are compared with analytical solutions and those available in the literature. Analysis of a realistic composite wind turbine blade is provided to show the capability of the present model for generalized composite slender structures. The analysis concludes that the proposed model can be used as a beam tool in a multibody framework whose valid range of rotation is up to $2\pi$.

I. Introduction

Many engineering components with one dimension larger than the other two can be idealized as beams, for example, bridges in civil engineering, joists and lever arms in heavy machine industries, and helicopter rotor blades in aeronautics. The present work is motivated by the need for higher-fidelity structural models in FAST, the National Renewable Energy Laboratory’s (NREL’s) premier aeroelastic computer-aided engineering (CAE) tool for wind turbines. The blades, tower, and shaft in a wind turbine system can be considered as beams. In weight-critical applications of beam structure such as wings in aerospace and blades in wind energy, composite materials are attractive due to their superior weight-to-strength and weight-to-stiffness ratios. The use of composites, however, complicates engineering analysis because of the coupling effects in the structure. Moreover, geometric nonlinearity is crucial to the analysis of highly flexible composite beam structures, especially when applied to unprecedented-length turbine blades.

The intrinsic formulation of geometric exact beam theory was first proposed by Reissner in 1973. Here the “intrinsic formulation” indicates that the one-dimensional (1D) strains are developed in terms of virtual displacement and virtual rotation quantities so that the formulation is not tied to a specific choice of displacement or rotation variables. Simo and Simo together with Vu-Quoc extended Reissner’s work to handle three-dimensional (3D) dynamic problems. Since then, researchers have reported many extensions and applications of GEBT, and huge effort has been invested in dealing with finite rotations. Jelenić and Crisfield implemented this theory based on finite element method, and introduced a new approach for interpolating the rotation field was introduced to preserve the geometric exactness. Betsch and Steinmann circumvented the interpolation of rotation by introducing a reparameterization of the weak form corresponding to the equations of motion of GEBT. Note that Ibrahimbegović implemented this theory by considering the initial twist and curvatures. More details on the vector-like parameterization of 3D finite rotations of this work are presented in Ref. 10. Ibrahimbegović and Mikdad then extended the previous static implementation to include dynamics. A brief review on the geometric exact beam theory and its implementation is found in Ref. 12. In contrast to the displacement-based implementations, the geometric exact beam theory also has been formulated by mixed finite elements where both the primary and dual field are interpolated independently. In the mixed formulation, all the necessary ingredients including Hamilton’s principle and the kinematic equations are combined in a single variational formulation statement with Lagrange multipliers. The motion variables, generalized strains, forces and moments, linear and angular momenta, and displacement and rotation variables are considered as independent quantities. Yu and Blair presented the implementation of GEBT in a mixed formulation where Rodrigues parameters are
chosen to represent the finite rotation. The work proposed a new time-marching scheme that is more efficient than that
developed by Patil et al.\textsuperscript{15} Yu and Blair’s work resulted in an open source general-purpose composite beam solver that
is called “GEBT”. Readers are referred to a textbook by Hodges, Nonlinear Composite Beam Theory,\textsuperscript{4} which contains
comprehensive derivations and discussions on nonlinear composite beam theories.

The objective of this paper is twofold: (1) implement a geometric exact beam theory with a Wiener-Milenković
parameter based on recent work of Yu and Blair,\textsuperscript{14} and (2) demonstrate the application of this approach in analysis
of composite beams. The authors expect that this paper will clear the path for wider and easier use of GEBT in
dynamic analysis and in a modularized multibody dynamics framework where the Wiener-Milenković parameters
are used.\textsuperscript{16,17} The next section recapitulates the Euler’s rotation theorem and the rotation tensor, and introduces
the concept of vectorial parameterization of finite rotation. The equations of geometric exact beam theory and the mixed
formulation using Wiener-Milenković parameter are discussed in Section 3. Section 4 presents a number of numerical
examples, including static, dynamic, and eigenvalue analyses of isotropic and composite beams, which validate the
present theory and illustrate the capability of this beam solver.

\section{II. Vectorial Parameterization of Rotation}

This section reviews the theories of rotation tensor and vectorial parameterization. The content of this section also
can be found in many other papers and textbooks.

\subsection{A. Euler’s Rotation Theorem and Rotation Tensor}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{rotation.png}
\caption{A finite rotation of magnitude $\phi$ about $\mathbf{n}$.}
\end{figure}

Euler’s rotation theorem states that an arbitrary motion of a rigid body that leaves one of its point fixed can be
represented by a single rotation of magnitude $\phi$ about unit vector $\mathbf{n}$.\textsuperscript{17} Figure 1 shows the configuration of the problem
where vector $\mathbf{a}$ rotates magnitude $\phi$ about unit vector $\mathbf{n}$; and the new vector after rotation is denoted as $\mathbf{b}$. In this section,
$\langle \cdot \rangle$ denotes a unit vector, and a single underline and double underlines denote a vector and a tensor, respectively. These
two vectors are related as
\begin{equation}
\mathbf{b} = \mathbf{C}^T \mathbf{a}
\end{equation}
where $\mathbf{C}$ is the rotation tensor and a superscript $T$ denotes the transpose. The fundamental property of a rotation
operation is to preserve length; vector length does not change when computed from its components resolved in different
arbitrary orthonormal bases. For example, $||\mathbf{a}|| = ||\mathbf{b}||$, where $|| \cdot ||$ denotes the Euclidean vector norm, which leads to
\begin{align}
\mathbf{C}^T \mathbf{a} &= ||\mathbf{a}|| \cos \alpha = ||\mathbf{b}|| \cos \alpha \\
||\mathbf{C} \mathbf{a}|| &= ||\mathbf{a}|| \sin \alpha = ||\mathbf{b}|| \sin \alpha
\end{align}
The tilde operator ($\tilde{\cdot}$) defines a second-order, skew-symmetric tensor corresponding to the given vector. In the litera-
ture, it is also termed as “cross-product matrix”. For example,
\begin{equation}
\tilde{\mathbf{n}} = \begin{bmatrix}
0 & -n_3 & n_2 \\
\ hline
n_3 & 0 & -n_1 \\
-n_2 & n_1 & 0
\end{bmatrix}
\end{equation}
The following equations can be drawn from the geometry depicted in Figure 1,

\[ \vec{b} = \vec{OC} + \vec{CB} \]
\[ = ||\vec{b}|| \cos \alpha \hat{n} + ||\hat{n}|| \sin \alpha [\hat{s} \cos \phi + \hat{\ell} \sin \phi] \]  

(4)

Unit vector \( \hat{\ell} \) is along the vector product of vectors \( \hat{n} \) and \( \hat{a} \),

\[ \hat{\ell} = \frac{\hat{n} \hat{a}}{||\hat{n}||} \]  

(5)

and unit vector \( \hat{s} \) is

\[ \hat{s} = \hat{\ell} \hat{n} = \left( \frac{\hat{n} \hat{a}}{||\hat{n}||} \right) \hat{n} \]  

(6)

With the help of Equation (2), Equation (3), Equation (5), and Equation (6), Equation (4) can be written as

\[ \vec{b} = \vec{a} + \sin \phi (\hat{n} \hat{a}) + (1 - \cos \phi) \hat{n} \hat{a} = \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} \hat{a} \end{bmatrix} \]  

(7)

where the rotation tensor \( C \) is

\[ C = A - \sin \phi \hat{n} + (1 - \cos \phi) \hat{n} \hat{a} \]  

(8)

with \( A \) a 3 \times 3 identity matrix. This equation is known as Rodrigues rotation formula.

### B. Wiener-Milenković Parameters

A finite rotation in 3D space discussed in the previous section leads to a one-to-one vector transformation, although four parameters (rotation magnitude \( \phi \) and rotation axis \( \hat{n} = \{n_1, n_2, n_3\}^T \)) are used instead of three. To eliminate the redundancy in the four-parameter description and to exploit the tensorial nature of rotation, we introduce the following definition.

\[ \vec{c} = p(\phi) \hat{n} \]  

(9)

where \( \vec{c} \) is the rotation parameter vector, \( p(\phi) \) is the generating function. For the Wiener-Milenković parameters used in the present paper,

\[ p(\phi) = 4 \tan \frac{\phi}{4} \]  

(10)

A singularity can be observed in this representation since \( ||\vec{c}|| \rightarrow \infty \) when \( |\phi| \rightarrow 2\pi \).

Substituting Eq. (9) into (8), the explicit expression of the rotation tensor in term of the vectorial parameterization can be obtained as

\[ C = A - \frac{\sin \phi}{p(\phi)} \hat{c} + \frac{1 - \cos \phi}{p^2(\phi)} \hat{c} \hat{c} \]  

(11)

For the Wiener-Milenković parameters, the rotation tensor is found by substituting Eq. (10) into (11) as

\[ C(\vec{c}) = \frac{1}{(4 - c_0)^2} \left[ (c_0^2 - \vec{c}^T \hat{c}) A - 2c_0 \hat{c} + 2c \hat{c}^T \right] \]  

(12)

where \( c_0 \) is a scalar parameter defined as

\[ c_0 = 2 \left( 1 - \tan^2 \frac{c}{2} \right) = 2 - \frac{\vec{c}^T \vec{c}}{8} \]  

(13)

In expanded form,

\[ C(\vec{c}) = \frac{1}{(4 - c_0)^2} \begin{bmatrix} c_0^2 + c_1^2 - c_2^2 - c_3^2 & 2(c_1 c_2 + c_0 c_3) & 2(c_1 c_3 - c_0 c_2) \\ 2(c_1 c_2 - c_0 c_3) & c_0^2 - c_1^2 + c_2^2 - c_3^2 & 2(c_2 c_3 + c_0 c_1) \\ 2(c_1 c_3 + c_0 c_2) & 2(c_2 c_3 - c_0 c_1) & c_0^2 - c_1^2 - c_2^2 + c_3^2 \end{bmatrix} \]  

(14)

where \( c_1, c_2, \) and \( c_3 \) are the components of vector \( \vec{c} \). For a linear theory, \( c_i \) are assumed to be small and their powers and products are negligible. The rotation matrix then reduces to

\[ C(\vec{c}) = A - \hat{c} \]  

(15)

For more details about the vectorial parameterization of finite rotations, interested readers are referred to a recent textbook, Bauchau, Flexible Multibody Dynamics, which discusses many aspects of multibody analysis.
III. Mixed Formulation of Geometric Exact Beam Theory

In this section, we derive the equations used in a mixed formulation of GEBT where finite rotation is represented by the Wiener-Milenković parameters. Figure 2 shows a beam in its undeformed and deformed states. A reference frame $b_i$ is introduced along the beam axis for the undeformed state; a frame $B_i$ is introduced along each point of the deformed beam axis. The variational statement of the geometric exact beam theory is\(^4\)

\[
\int_{t_1}^{t_2} \int_0^l \left[ \delta \nu^T P + \delta \Omega^T H - \delta \gamma^T F - \delta \kappa^T M + \delta \bar{q}^T f + \delta \bar{\psi}^T m \right] dx_1 dt
\]

\[
= \int_0^l \left( \delta \nu^T \dot{P} + \delta \bar{\psi}^T \dot{H} \right) \bigg|_{t_1}^{t_2} dx_1 - \int_0^l \left( \delta \nu^T F + \delta \bar{\psi}^T M \right) \bigg|_{t_1}^{t_2} dt
\]  

(16)

where $\bar{q}$ and $\bar{\psi}$ are the virtual displacement and rotation, respectively; $F$ and $M$ are the sectional force resultant and moment resultant, respectively; $P$ and $H$ are the sectional linear and angular momentum, respectively; $V$ and $\Omega$ are the linear and angular velocities of the beam reference line, respectively; $\gamma$ and $\kappa$ are the force-strain measures and moment-strain measures, respectively; $f$ and $m$ are the distributed forces and moments per unit length, respectively; $\dot{F}$ and $\dot{M}$ are the forces and moments, respectively, evaluated at the ends of space interval; $\dot{P}$ and $\dot{H}$ are the linear momentum and angular momentum, respectively, evaluated at the ends of time interval. Note that unlike in the previous section the overbarred symbols, $\bar{\cdot}$, denote virtual quantities in this section. Using the one-dimensional virtual kinematic relations\(^4\) and integrating this equation by parts with respect to $t$ and $x_1$, we obtain the following

\[
\int_{t_1}^{t_2} \int_0^l \left\{ \delta \nu^T \left( F' + \hat{K} F - \hat{P} - \hat{\Omega} P + f \right) \\
+ \delta \bar{\psi}^T \left[ M' + \hat{K} M + (\hat{\kappa}_1 + \hat{\bar{\psi}}) F - \hat{V} P - \hat{H} \Omega + m \right] \right\} dx_1 dt
\]

\[
= \int_0^l \left( \delta \nu^T \dot{P} + \delta \bar{\psi}^T \dot{H} \right) \bigg|_{t_1}^{t_2} dx_1 - \int_0^l \left( \delta \nu^T F + \delta \bar{\psi}^T M \right) \bigg|_{t_1}^{t_2} dt
\]  

(17)

where $K = \kappa + k$ is the summation of moment-strain measures and initial curvatures; $\hat{\kappa}_1$ is defined as a column as $e_1 = [1 \ 0 \ 0]^T$. The check operator on the right-hand-side of the equation is defined as $\hat{\cdot} = (\cdot) - (\cdot)$. The “primed” and “dotted” terms represent their spatial and temporal partial derivatives, respectively. Note that all the quantities in Equation (16) and Equation (17) are expressed in the deformed base $B$. The constitutive equations relate the velocities to the momenta and the one-dimensional strain measures to the sectional resultants as

\[
\begin{bmatrix} P \\ H \end{bmatrix} = I \begin{bmatrix} V \\ \Omega \end{bmatrix}
\]

(18)

\[
\begin{bmatrix} F \\ M \end{bmatrix} = S \begin{bmatrix} \gamma \\ \kappa \end{bmatrix}
\]

(19)

where $I$ and $S$ are the sectional mass and stiffness matrices, respectively. For composite beam structures, the sectional stiffness matrix $S$ could be fully populated, which means that all of the fundamental deformation modes, including extension, shear, torsion, and bending, could be coupled.

To derive the mixed formulation, the kinematical relations are incorporated into the original variational statement of Hamilton’s principle to obtain a new functional so that all of the variables can vary independently in the calculus of variations. The one-dimensional kinematical equations are written as follows.

\[
u' = C^{bB}(e_1 + \gamma) - e_1 - \hat{k}u
\]

(20)

\[
u = C^{bB}V - v - \hat{\omega}u
\]

(21)

\[
u' = Q^{-1}(\kappa + k - C^{Bb}k)
\]

(22)

\[
u = Q^{-1}(\Omega - C^{Bb})\omega
\]

(23)

where $u$ is the one-dimensional displacement; $C^{bB}$ is the rotation matrix and $C^{Bb} = (C^{bB})^T$; $v$ and $\omega$ are the linear and angular velocities of undeformed triad $b$ in the inertial frame; $c$ are the Wiener-Milenković parameters introduced in the previous section; $Q$ is a matrix defined as follows.

\[
Q = \frac{1}{(4 - c_6)^2} \left[ (4 - \frac{1}{4} c^T c) \Delta - 2\hat{c} + \frac{1}{2} c c^T \right]
\]

(24)

and

\[
Q^{-1} = (1 - \frac{1}{16} c^T c) \Delta + \frac{1}{2} \hat{c} + \frac{1}{8} c c^T
\]

(25)

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The Lagrange multipliers, \( \lambda_i \), can be identified by using the condition that virtual quantities can be independently and arbitrarily varied.

To deal with moving beams, we introduce a global body-attached coordinate system \( \alpha \), and it is clear that the relation \( C^{ab} \) between frame \( a \) and undeformed frame \( b \) is determined and time independent. If we introduce another set of Wiener-Milenković parameters \( c_a \) which is defined in frame \( a \), then the following equation can be derived

\[
C^{Ba} = C^{ba} C
\]  
(27)

Here, \( C \) a function of \( c_a \) identical to \( C^{Bb} \) in Eq. (14) where \( c \) is replaced by \( c_a \). After identifying the Lagrange multipliers in Eq. (26), the variational statement in Eq. (17) can be rewritten as follows.

\[
\int_{t_1}^{t} \left\{ \delta V^T P + \delta \Omega^T H - \delta \gamma^T F - \delta \kappa^T M + \delta q^T f + \delta \psi^T m \right\} dt
\]

\[
+ \delta \left[ \lambda_1 \left( u' - C^{bB}(\epsilon_1 + \gamma) + e_1 + \bar{k}u \right) \right] \\
+ \delta \left[ \lambda_2 \left( \bar{\dot{u}} - C^{bB} V + v + \bar{\omega}u \right) \right] \\
+ \delta \left[ \lambda_3 \left( \epsilon' - Q^{-1}(\kappa + k - C^{Bb} k)u \right) \right] \\
+ \delta \left[ \lambda_4 \left( \dot{\epsilon} - Q^{-1}(\Omega - C^{Bb}) \right) \right] dx_1 dt
\]  
(26)

These kinematical equations are adjoined to Eq. (16) by Lagrange multipliers so that the left-hand-side becomes:

\[
\int_{t_1}^{t} \left\{ \delta V^T P + \delta \Omega^T H - \delta \gamma^T F - \delta \kappa^T M + \delta q^T f + \delta \psi^T m \right\} dt
\]

\[
+ \delta \left[ \lambda_1 \left( u' - C^{bB}(\epsilon_1 + \gamma) + e_1 + \bar{k}u \right) \right] \\
+ \delta \left[ \lambda_2 \left( \bar{\dot{u}} - C^{bB} V + v + \bar{\omega}u \right) \right] \\
+ \delta \left[ \lambda_3 \left( \epsilon' - Q^{-1}(\kappa + k - C^{Bb} k)u \right) \right] \\
+ \delta \left[ \lambda_4 \left( \dot{\epsilon} - Q^{-1}(\Omega - C^{Bb}) \right) \right] dx_1 dt
\]  
(27)

Note that the time derivatives of virtual quantities are removed by carrying out integration by parts and \( Q_a \) is defined similar to Eq. (27) as

\[
Q_a = C^{ab} Q C^{ba}
\]  
(29)

The subscripts in Eq. (28) indicate in which frame the quantities are defined, and they can be easily transformed between different frames using the rotation matrix.

Equation (28) contains all the information needed for the mixed formulation of geometric exact beam theory. We consider \( u_a, c_a, F_B, M_B, P_B, \) and \( H_B \) as the fundamental unknown variables. Linear shape functions are used for test
functions $\delta u_a$, $\delta \psi_a$, $\delta F_a$, and $\delta M_a$ because only the first order spatial derivatives appear in these terms, and constant shape functions are used for $\delta F_a$ and $\delta M_a$.

Dividing a beam into $N$ elements with the starting node of the $i$ element denoted as $i$ and the ending number as $i + 1$, and using the linear and constant shape functions for different variables, the integration in Eq. (28) can be calculated analytically. Finally, we conclude the following finite element equations:

$$\begin{align*}
    f_{u_i}^- - F_i^* &= 0 \\
    f_{\psi_i}^- - M_i^* &= 0 \\
    f_{F_i}^- - \ddot{u}_i &= 0 \\
    f_{M_i}^- - \dddot{c}_i &= 0
\end{align*}$$

at the starting node, and

$$\begin{align*}
    f_{u_N}^+ - F_{N+1}^* &= 0 \\
    f_{\psi_N}^+ - M_{N+1}^* &= 0 \\
    f_{F_N}^+ + \ddot{u}_{N+1} &= 0 \\
    f_{M_N}^+ + \dddot{c}_{N+1} &= 0
\end{align*}$$

at the ending node. Note $F_i^*, M_i^*, F_{N+1}^*$, and $M_{N+1}^*$ are the external forces/moments balancing the internal resultants.

At each intermediate point the equations are

$$\begin{align*}
    f_{u_i}^+ + f_{u_{i+1}^-} &= 0 \\
    f_{\psi_i}^+ + f_{\psi_{i+1}^-} &= 0 \\
    f_{F_i}^+ + f_{F_{i+1}^-} &= 0 \\
    f_{M_i}^+ + f_{M_{i+1}^-} &= 0
\end{align*}$$

for $i = 1, 2, \ldots, N - 1$. Also for each element for $i = 1, \ldots, N$, we have the following:

$$\begin{align*}
    f_{P_i} &= 0 \\
    f_{H_i} &= 0
\end{align*}$$

The $f$ matrices in the above equations are calculated analytically from the integration in Equation (28) and are written as

$$\begin{align*}
    f_{u_i}^\pm &= \mp C^T C^{ab} F_i - \bar{f}_i^\pm + \frac{\Delta L_i}{2} \left[ \bar{\omega}_a C^T C^{ab} P_i + \bar{C}^T \bar{C}^{ab} P_i \right] \\
    f_{\psi_i}^\pm &= \mp C^T C^{ab} M_i - \bar{m}_i^\pm + \frac{\Delta L_i}{2} \left[ \bar{\omega}_a C^T C^{ab} H_i + \bar{C}^T \bar{C}^{ab} H_i + C^T C^{ab} (\bar{V}_i P_i - (\dddot{c}_i + \dddot{\gamma}_i)) F_i \right] \\
    f_{F_i}^\pm &= \pm u_i - \frac{\Delta L_i}{2} \left[ C^T C^{ab} (e_i + \gamma_i) - C^{ab} e_i \right] \\
    f_{M_i}^\pm &= \pm c_i - \frac{\Delta L_i}{2} Q_a^{-1} C^{ab} \kappa_i \\
    f_{P_i} &= C^T C^{ab} V_i - v_i - \bar{\omega}_a u_i - \dddot{u}_i \\
    f_{H_i} &= \Omega_i - C^{ba} C \omega_a - C_{ba} Q_a \dddot{c}_i
\end{align*}$$

with

$$\begin{align*}
    \bar{f}_i^- &= \int_0^1 (1 - \eta) f_a \Delta L_i d\eta \\
    \bar{f}_i^+ &= \int_0^1 \eta f_a \Delta L_i d\eta \\
    \bar{m}_i^- &= \int_0^1 (1 - \eta) m_a \Delta L_i d\eta \\
    \bar{m}_i^+ &= \int_0^1 \eta m_a \Delta L_i d\eta
\end{align*}$$

where $\Delta L_i$ is the length the $i^{th}$ element; $L_i$ is the $x_1$ coordinate of the starting node; and $\eta$ is a general coordinate defined as shown below:

$$\eta = \frac{x_1 - L_i}{\Delta L_i}$$

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Table 1: Tip Displacements of the Cantilever Beam Subject to a Constant Moment (in Inches)

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>Analytical Solution</th>
<th>GEBT</th>
<th>Number of Required Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>([-2.9181, 6.5984])</td>
<td>([-2.9161, 6.6004])</td>
<td>9</td>
</tr>
<tr>
<td>0.8</td>
<td>([-9.1935, 8.6374])</td>
<td>([-9.1946, 8.6455])</td>
<td>14</td>
</tr>
<tr>
<td>1.2</td>
<td>([-13.8710, 5.7583])</td>
<td>([-13.8800, 5.7576])</td>
<td>19</td>
</tr>
<tr>
<td>1.6</td>
<td>([-14.2705, 1.6496])</td>
<td>([-14.2403, 1.6287])</td>
<td>25</td>
</tr>
<tr>
<td>2.0</td>
<td>([-12.0, 0.0])</td>
<td>([-11.9017, -0.0709])</td>
<td>968</td>
</tr>
</tbody>
</table>

In the governing equations from Equation (30) to Equation (43), those corresponding to \( f_u \) and \( f_\psi \) are the equations of motion; the equations corresponding to \( f_F \) and \( f_M \) are the strain-displacement equations; and the equations corresponding to \( f_P \) and \( f_H \) are the velocity-displacement kinematical equations. These equations can also be written in the symbolic form as

\[
\mathbf{F}(\mathbf{X}, \dot{\mathbf{X}}) = 0
\]  

(55)

where \( \mathbf{F} \) is a system of \( 18N + 6M \) equations and \( \mathbf{X} \) is a vector containing \( 18N + 6M \) unknowns. Here \( N \) is the number of elements and \( M \) is the total number of boundary points and connection points in beam or beam assemblies.

The system of nonlinear equations is solved using Newton-Raphson method along with line search to guarantee global convergence. A Newmark-type time-marching scheme is derived for transient analysis. For the eigenvalue analysis, GEBT calculates the steady state solution first, and the eigenvalue analysis is performed corresponding to this state. Readers can refer to Yu and Blair\textsuperscript{14} for more details on eigenvalue solvers and time-marching schemes in solving these equations.

IV. Numerical Examples

A. Example 1: Static Bending of a Cantilever Beam

The first example is a benchmark problem for geometrically nonlinear analysis of beams.\textsuperscript{5,19} We calculate the static deflection of a cantilever beam that is subjected at its free end to a constant moment \( M \). The length of the beam \( L \) is 12 in and the side length of the square section is 1 in. The beam is discretized into 16 elements in the GEBT calculation to achieve a converged solution. The Young’s modulus \( E \) of the material is given as \( 30 \times 10^6 \) lb/in\(^2\). The moment applied to the beam is \( M = \lambda \bar{M} \) where \( \bar{M} = \pi \frac{EI}{L^3} \). The parameter \( \lambda \) will vary between 0 and 2. Here, the tip displacements are compared with the analytical solutions, which can be found in Mayo et al.\textsuperscript{20}. Table 1 lists the analytical solution and the results obtained by the current GEBT calculation. Good agreement between these two sets of results is observed. It is noted that as the applied moment is increased, the number of iterations needed to obtain a converged results in GEBT increases, and a singularity exists when the rotation angle reaches \( 2\pi \). The GEBT results in the last line of in Table 1 is approximated by given \( \lambda \) as 1.96. The deformation of the beam under different applied bending moment is presented in Figure 3.

Figure 3: Calculated Deformation of a Cantilever Beam of Example 1 Under Several Constant Bending Moments

![Figure 3: Calculated Deformation of a Cantilever Beam of Example 1 Under Several Constant Bending Moments](image-url)
Table 2: Tip Displacements of a Curved Beam Subject to a Concentrated Load (in Inches)

<table>
<thead>
<tr>
<th></th>
<th>Bathe-Bolourchi$^{21}$</th>
<th>GEBT</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_1$</td>
<td>-13.4</td>
<td>-13.4</td>
</tr>
<tr>
<td>$U_2$</td>
<td>-23.5</td>
<td>-23.5</td>
</tr>
<tr>
<td>$U_3$</td>
<td>53.4</td>
<td>53.4</td>
</tr>
</tbody>
</table>

B. Example 2: Static Bending of a Curved Beam in Three-Dimensional Space

The second example is to validate the capability of GEBT for initially curved beams. First calculate the static deflection of a 45°-bend cantilever beam subjected to a concentrated dead end load, which was proposed by Bathe and Bolourchi$^{21}$ and is used widely as a benchmark for curved-beam analysis. The beam lies in the $x-y$ plane and the tip load is applied along the $z$ direction with the magnitude $P = 600$ lb. The radius of curvature is 100 in and the material properties are given as $E = 10^7$ psi and $\nu = 0.0$. The side length of the square cross-section of the beam is 1.0 in. A sketch of the problem is presented in Figure 4. In the GEBT calculation, the beam is discretized as 16 elements. The tip displacements are provided in Table 2 for comparison; good agreement is demonstrated.

Figure 4: Schematic of the Undeformed Curved Beam Used in Example 2

C. Example 3: Dynamic Analysis of a Beam Assembly

The capability of analyzing beam assemblies and dynamic behavior of GEBT is tested in this example. A joined-beam model with two cantilever beam members meeting at their tips is used for the analysis$^{22}$. The sketch of this beam assembly is presented in Figure 5 and the cross-section of the cantilever is rectangular with width 0.1 m and thickness 0.05 m. The material properties of the beam are given as: $E = 70$ GPa, $\nu = 0.35$, and $\rho = 2700$ kg/m$^3$. A sinusoidal vertical force is applied at the joint of the beam assembly, given by

$$F_z(t) = \begin{cases} 
0 & t < 0 \\
A_F \sin(\omega_F t) & t \geq 0 
\end{cases} \tag{56}$$

with $A_F = 1.0 \times 10^5$ N and $\omega_F = 20$ rad/s. Each beam member is discretized into 10 elements. The results obtained by GEBT are compared with those from an ANSYS calculation. BEAM 188 elements are used and the mesh is the same as that used in GEBT. The time steps are 0.001 s in GEBT and ANSYS. The non-zero displacement components of the joint are plotted in Figure 6 and good agreement is shown. Because the rotations are described by the rotation parameters in the present work, it is easy to deal with another important case in structural analysis, the structure with follower force whose direction is changing along with the deformation of the structure. Figure 7 shows the behavior of this beam assembly under a dead force and a follower force. It is apparent that the structure with a follower force is softer than that with dead force for nonlinear analysis.

Finally, an eigenvalue analysis was conducted using GEBT and ANSYS for validation. The lowest five natural frequencies are listed in Table 3. The percent differences between the results are calculated as $\frac{||\text{GEBT} - \text{ANSYS}||}{\text{ANSYS}} \times 100$. Again, good agreement between the results obtained by GEBT and ANSYS is demonstrated, where the largest error is only 1.57%.
Figure 5: Schematic of the Undeformed Joint Beam Assembly Used in Example 3

Figure 6: ANSYS and GEBT Calculated Tip Displacement Histories of the Joined Beam Under Vertical Tip Load
Figure 7: GEBT Calculated Tip Displacement Histories of the Joined Beam Under Dead and Follower Vertical Tip Loads

Table 3: Comparison of Natural Frequencies of Beam Assembly in Example 3 (in Hz)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>GEBT</td>
<td>35.23</td>
<td>171.02</td>
<td>215.87</td>
<td>275.05</td>
<td>407.2</td>
</tr>
<tr>
<td>ANSYS</td>
<td>35.17</td>
<td>169.83</td>
<td>212.53</td>
<td>273.34</td>
<td>403.53</td>
</tr>
<tr>
<td>Percent Difference</td>
<td>0.17</td>
<td>0.70</td>
<td>1.57</td>
<td>0.63</td>
<td>0.91</td>
</tr>
</tbody>
</table>
D. Example 4: Dynamic Analysis of a Wind Turbine Blade

The last example is a transient analysis of a composite beam with cross-section that is representative of a wind turbine blade, although here it is constant along the length. An MH 104 airfoil, which was studied by Chen et al.,\textsuperscript{23} is used in this case. The blade is 60 m long and cantilevered at one end. The sectional properties, including material and geometric constants, and units can be found in Chen et al.\textsuperscript{23} and are given below

\[
I = \begin{bmatrix}
258.053 & 0.00 & 0.00 & 7.07839 & -71.6871 \\
258.053 & 0.00 & -7.07839 & 0.00 & 0.00 \\
258.053 & 71.6871 & 0.00 & 0.00 & 0.00 \\
48.59 & 0.00 & 0.00 & 0.00 & 0.00 \\
Symmetry & 2.172 & 0.00 & 0.00 & 46.418 \\
\end{bmatrix}
\]  

Note that the sectional stiffness matrix is fully populated in this case, which means full elastic coupling between extension, shear, twist and bending are taken into consideration. The beam is divided into 10 elements in the GEBT calculation and a sinusoidal point force governed by Equation (56) is applied at the free tip in the \( z \) direction with \( A_F = 1.0 \times 10^5 \) N and \( \omega_F = 20 \) rad/s (see Figure 8). A sketch of this example is shown in Figure 9. The time step used in this example is 0.001 s so that a set of converged results can be achieved. The tip displacement and rotation histories of the beam are plotted in Figure 10. Note that all the components, including three displacements and three rotations, are non-zero due to the coupling effects. The time histories of the stress resultants at the root of the beam are given in Figure 11.

\[
S = \begin{bmatrix}
4.334E+08 & -3.741E+06 & -2.935E+07 & 1.527E+07 & 3.835E+05 & -4.742E+06 \\
2.743E+07 & -4.592E+04 & -6.869E+04 & -4.742E+06 & 1.430E+06 & 4.406E+08 \\
\end{bmatrix}
\]  

Figure 8: The Sinusoidal Vertical Force Used in Example 4

Figure 9: Schematic of the Composite Wind Turbine Blade Under Vertical Force in Example 4
Figure 10: Tip Displacement and Rotation Histories of a Realistic Wind Turbine Blade Under Vertical Tip Load
Figure 11: Stress-resultant Time Histories At the Root of a Wind Turbine Blade
V. Conclusion

This paper presents an implementation of geometric exact beam theory. Based on previous work,\textsuperscript{14} the Wiener-Milenković parameter is used to describe rotation in three-dimensional space. The valid range of rotation is extended, however, due to the analytical nature of the mixed formulation, a singularity occurs when the rotation reaches $2\pi$. Numerical examples that demonstrate the capability of the present beam solver are presented. Two benchmark static problems for geometric nonlinear and curved beams are examined. The agreement between the results calculated by the proposed model and the analytical solution and those are available in the literature are excellent. A dynamic analysis of a beam assembly was conducted in GEBT and ANSYS, and GEBT showed good agreement with ANSYS for displacements and natural frequencies. Finally, a composite beam with realistic wind turbine cross-section was analyzed, and all coupling effects were accounted for, including those between extension, shear, bending, and torsion. The study concludes that GEBT is a powerful tool for composite beam analysis that can be used as a module in the multibody analysis framework.

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References

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